Solving G-Cubed Models without policy optimisation

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Abstract

This paper sets out the algebra involved in solving the G-Cubed class of macroeconomic models, without allowing for optimal policy. Policy responses are instead captured by exogenously determined policy response functions. It provides the line-by-line algebraic details needed to implement the two-point boundary solution algorithm described in McKibbin (1987) that builds on the work of Blanchard and Kahn (1980).

1 The non-linear model

The model can be expressed in terms of the following variables:

- the vector of state variables, S_t;
- the vector of costate or "jump" variables, J_t;
- the vector, R_t, the subset of endogenous variables that also enter the model in expectations, at time, t of their value in time $t+1$, $E_t(R_{t+1})$;
- the vector of other endogenous variables, Z_t; and
- the vector of exogenous variables, X_t.

In non-linear form, these equations can be written:

$$S_{t+1} = \Phi_S\left(S_{t+1}, E_t(J_{t+1}), R_t, Z_t, E_t(R_{t+1}), S_t, J_t, X_t\right) \tag{1}$$
$$E_t(J_{t+1}) = \Phi_J\left(S_{t+1}, E_t(J_{t+1}), R_t, Z_t, E_t(R_{t+1}), S_t, J_t, X_t\right) \tag{2}$$
$$R_t = \Phi_R\left(S_{t+1}, E_t(J_{t+1}), R_t, Z_t, E_t(R_{t+1}), S_t, J_t, X_t\right) \tag{3}$$
$$Z_t = \Phi_Z\left(S_{t+1}, E_t(J_{t+1}), R_t, Z_t, E_t(R_{t+1}), S_t, J_t, X_t\right) \tag{4}$$

2 The linear model as a series of differential equations

The model is linearised using a first-order Taylor-series expansion around a point P:

$$P = (\bar{S}_{t+1}, \bar{S}_t, E_t(\bar{J}_{t+1}), \bar{J}_t, \bar{R}_t, E_t(\bar{R}_{t+1}), \bar{Z}_t, \bar{X}_t)$$

In most cases, P is defined as values for the variables in the year prior to the start of model projections.

Adjustments are made to the values selected for variables that are a long way from their equilibrium values, for example, when real interest rates are negative, those negative interest rates will not be used as part of the point around which the model is linearised.

Calculating numerical derivatives at the linearisation point, the model can be approximated by a set of linear differential equations. For A in (S, J, R, Z), define ϕ_{An} as the matrix of partial derivatives of function $\Phi_A()$ with respect to its nth argument.

To improve readability, define the following notation in time-period t:

$$\begin{aligned} dS_t &= s_t \\ dJ_t &= j_t \\ dR_t &= r_t \\ dZ_t &= z_t \\ dX_t &= x_t \\ E_t(r_{t+1}) &= {}_t r_{t+1} \\ E_t(j_{t+1}) &= {}_t j_{t+1} \end{aligned} \tag{5}$$

The linear model is then the following differential equations.

$$s_{t+1} = \phi_{s1} s_{t+1} + \phi_{s2}\ {}_t j_{t+1} + \phi_{s3} r_t + \phi_{s4} z_t + \phi_{s5}\ {}_t r_{t+1} + \phi_{s6} s_t + \phi_{s7} j_t + \phi_{s8} x_t \tag{6}$$

$${}_t j_{t+1} = \phi_{j1} s_{t+1} + \phi_{j2}\ {}_t j_{t+1} + \phi_{j3} r_t + \phi_{j4} z_t + \phi_{j5}\ {}_t r_{t+1} + \phi_{j6} s_t + \phi_{j7} j_t + \phi_{j8} x_t \tag{7}$$

$$r_t = \phi_{r1} s_{t+1} + \phi_{r2}\ {}_t j_{t+1} + \phi_{r3} r_t + \phi_{r4} z_t + \phi_{r5}\ {}_t r_{t+1} + \phi_{r6} s_t + \phi_{r7} j_t + \phi_{r8} x_t \tag{8}$$

$$z_t = \phi_{z1} s_{t+1} + \phi_{z2}\ {}_t j_{t+1} + \phi_{z3} r_t + \phi_{z4} z_t + \phi_{z5}\ {}_t r_{t+1} + \phi_{z6} s_t + \phi_{z7} j_t + \phi_{z8} x_t \tag{9}$$

3 The State-Space Form

The State-Space Form (SSF) is obtained by eliminating s_{t+1}, ${}_t j_{t+1}$, r_t, and z_t from the right-hand side of the linear model.

Working through the necessary steps, first rearrange equation 9 to gather all z_t terms on the left:

$$(I_z - \phi_{z4}) z_t = \phi_{z1} s_{t+1} + \phi_{z2}\ {}_t j_{t+1} + \phi_{z3} r_t + \phi_{z5}\ {}_t r_{t+1} + \phi_{z6} s_t + \phi_{z7} j_t + \phi_{z8} x_t \tag{10}$$

Multiplying through by the inverse of $(I_z - \phi_{z4})$ and defining the α coefficient matrices appropriately, we obtain:

$$z_t = \alpha_{z1} s_{t+1} + \alpha_{z2}\ {}_t j_{t+1} + \alpha_{z3} r_t + \alpha_{z5}\ {}_t r_{t+1} + \alpha_{z6} s_t + \alpha_{z7} j_t + \alpha_{z8} x_t \tag{11}$$

Use equation 11, to substitute z_t out of equations 6, 7, and 8, again defining α coefficient matrices appropriately:

$$s_{t+1} = \alpha_{s1}s_{t+1} + \alpha_{s2}\ {}_tj_{t+1} + \alpha_{s3}r_t + \alpha_{s5}\ {}_tr_{t+1} + \alpha_{s6}s_t + \alpha_{s7}j_t + \alpha_{s8}x_t \tag{12}$$

$${}_tj_{t+1} = \alpha_{j1}s_{t+1} + \alpha_{j2}\ {}_tj_{t+1} + \alpha_{j3}r_t + \alpha_{j5}\ {}_tr_{t+1} + \alpha_{j6}s_t + \alpha_{j7}j_t + \alpha_{j8}x_t \tag{13}$$

$$r_t = \alpha_{r1}s_{t+1} + \alpha_{r2}\ {}_tj_{t+1} + \alpha_{r3}r_t + \alpha_{r5}\ {}_tr_{t+1} + \alpha_{r6}s_t + \alpha_{r7}j_t + \alpha_{r8}x_t \tag{14}$$

Repeat the process to eliminate r_t from the right of the linear model, rearranging equation 14:

$$(I_r - \alpha_{r3})r_t = \alpha_{r1}s_{t+1} + \alpha_{r2}\ {}_tj_{t+1} + \alpha_{r4}z_t + \alpha_{r5}\ {}_tr_{t+1} + \alpha_{r6}s_t + \alpha_{r7}j_t + \alpha_{r8}x_t \tag{15}$$

Multiply through by the inverse of $(I_r - \alpha_{r3})$, to solve for r_t, defining the β coefficient matrices appropriately.

$$r_t = \beta_{r1}s_{t+1} + \beta_{r2}\ {}_tj_{t+1} + \beta_{r5}\ {}_tr_{t+1} + \beta_{r6}s_t + \beta_{r7}j_t + \beta_{r8}x_t \tag{16}$$

Use equation 16, to substitute r_t out of equations 12, 13, and 11, defining the β coefficient matrices appropriately.

$$s_{t+1} = \beta_{s1}s_{t+1} + \beta_{s2}\ {}_tj_{t+1} + \beta_{s5}\ {}_tr_{t+1} + \beta_{s6}s_t + \beta_{s7}j_t + \beta_{s8}x_t \tag{17}$$

$${}_tj_{t+1} = \beta_{j1}s_{t+1} + \beta_{j2}\ {}_tj_{t+1} + \beta_{j5}\ {}_tr_{t+1} + \beta_{j6}s_t + \beta_{j7}j_t + \beta_{j8}x_t \tag{18}$$

$$z_t = \beta_{z1}s_{t+1} + \beta_{z2}\ {}_tj_{t+1} + \beta_{z5}\ {}_tr_{t+1} + \beta_{z6}s_t + \beta_{z7}j_t + \beta_{z8}x_t \tag{19}$$

Repeat the process to eliminate ${}_tj_{t+1}$ from the right of the linear model, rearranging equation 18:

$$(I_j - \beta_{j2})\ {}_tj_{t+1} = \beta_{j1}s_{t+1} + \beta_{j5}\ {}_tr_{t+1} + \beta_{j6}s_t + \beta_{j7}j_t + \beta_{j8}x_t \tag{20}$$

Multiply through by the inverse of $(I_j - \beta_{j2})$ to solve for ${}_tj_{t+1}$, defining the γ coefficient matrices appropriately.

$$ {}_{t}j_{t+1} = \gamma_{j1} s_{t+1} + \gamma_{j5}\ {}_{t}r_{t+1} + \gamma_{j6} s_{t-1} + \gamma_{j7} j_t + \gamma_{j8} x_t \tag{21} $$

Use equation 21 to substitute ${}_{t}j_{t+1}$ out of equations 12, 14, and 11, defining the γ coefficient matrices appropriately:

$$ s_{t+1} = \gamma_{s1} s_{t+1} + \gamma_{s5}\ {}_{t}r_{t+1} + \gamma_{s6} s_t + \gamma_{s7} j_t + \gamma_{s8} x_t \tag{22} $$

$$ r_t = \gamma_{r1} s_{t+1} + \gamma_{r5}\ {}_{t}r_{t+1} + \gamma_{r6} s_t + \gamma_{r7} j_t + \gamma_{r8} x_t \tag{23} $$

$$ z_t = \gamma_{z1} s_{t+1} + \gamma_{z5}\ {}_{t}r_{t+1} + \gamma_{z6} s_t + \gamma_{z7} j_t + \gamma_{z8} x_t \tag{24} $$

Repeat the process to eliminate s_{t+1} from the right of the linear model, rearranging equation 22:

$$ (I_s - \gamma_{s1}) s_{t+1} = \gamma_{s5}\ {}_{t}r_{t+1} + \gamma_{s6} s_t + \gamma_{s7} j_t + \gamma_{s8} x_t \tag{25} $$

Multiply through by the inverse of $(I_s - \gamma_{s1})$, to solve for s_{t+1}, defining the δ coefficient matrices appropriately.

$$s_{t+1} = \delta_{sr}\ {}_tr_{t+1} + \delta_{ss}s_t + \delta_{sj}j_t + \delta_{sx}x_t \tag{26}$$

Use equation 26 to substitute s_{t+1} out of equations 21, 23, and 24, defining the δ coefficient matrices appropriately. Note that the δ coefficient matrices are referenced throughout the rest of this document. Their subscripts indicate the vectors that they relate, with the first subscript indicating the vector on the left of the equation and the second subscript indicating the vector on the right of the equation. Thus, δ_{sj} is the matrix of SSF coefficients describing the relationship between s_{t+1} on the left and j_t on the right.

$${}_tj_{t+1} = \delta_{jr}\ {}_tr_{t+1} + \delta_{js}s_t + \delta_{jj}j_t + \delta_{jx}x_t \tag{27}$$

$$r_t = \delta_{rr}\ {}_tr_{t+1} + \delta_{rs}s_t + \delta_{rj}j_t + \delta_{rx}x_t \tag{28}$$

$$z_t = \delta_{zr}\ {}_tr_{t+1} + \delta_{zs}s_t + \delta_{zj}j_t + \delta_{zx}x_t \tag{29}$$

Equations 26, 27, 28, and 29 constitute the SSF.

4 The stable manifold

The stable manifold is the non-exploding path for the costate variables, j_t. It is found using the iterative algorithm described in McKibbin (1987).

4.1 The first iteration to find the stable manifold

The model needs to be solved for paths for all variables from period $t = 0$ to period $t = T$. Use a suitable terminal condition to start the iterative process of computing the transition rule for the costate variable that implies model consistent expectations.

With (linearised) model-consistent expectations, we can slightly simplify notation again, replacing ${}_tr_{t+1}$ and ${}_tj_{t+1}$with r_{t+1} and j_{t+1} respectively because the model has no sources of uncertainty.

The first iteration yields rules for j_T, r_T in terms of s_T and x_T. These rules must be model consistent with remaining periods. Being the final period, T, a terminal condition is required.

An intuitive terminal condition involves the period-to-period change in the variables being zero from period T to period $T + 1$ so $j_T = j_{T+1}$ and $r_T = r_{T+1}$. This enables elimination of r_t and j_t from the period T SSF equations.

First eliminate $r_{T+1} = r_T$ from the right of equation 28.

$$(I_r - \delta_{rr})\, r_T = \delta_{rs}s_T + \delta_{rj}j_T + \delta_{rx}x_T \tag{30}$$

Multiply through by $\Gamma_{rT} = (I_r - \delta_{rr})^{-1}$.

$$r_T = \psi_{rs} s_T + \psi_{rj} j_T + \psi_{rx} x_T \tag{31}$$

The coefficient matrices in equation 31 are defined as follows.

$$\psi_{rs} = \Gamma_{rT}\delta_{rs} \tag{32}$$
$$\psi_{rj} = \Gamma_{rT}\delta_{rj} \tag{33}$$
$$\psi_{rx} = \Gamma_{rT}\delta_{rx} \tag{34}$$

Using the terminal condition again, $r_{T+1} = r_T$ substitute equation 31 for r_{T+1} in equation 27.

$$\begin{aligned} j_{T+1} = \delta_{jr} \left(\psi_{rs} s_T + \psi_{rj} j_T + \psi_{rx} x_T\right) \\ + \delta_{js} s_T + \delta_{jj} j_T + \delta_{jx} x_T \end{aligned} \tag{35}$$

Given the terminal condition implies $j_{T+1} = j_T$, collect j_T terms on the left.

$$\begin{aligned} (I_j - \delta_{jj} - \delta_{jr}\psi_{rj}) j_T = \delta_{jr} \left(\psi_{rs} s_T + \psi_{rx} x_T\right) \\ + \delta_{js} s_T + \delta_{jx} x_T \end{aligned} \tag{36}$$

Multiply through by $\Gamma_{jT} = (I_j - \delta_{jj} - \delta_{jr}\psi_{rj})^{-1}$ to obtain a rule for j_T in terms of s_T and x_T.

$$j_T = H_{1T} s_T + H_{2T} x_T \tag{37}$$

The coefficient matrices in equation 37 are defined as follows.

$$H_{1T} = \Gamma_{jT} \left(\delta_{jr}\psi_{rs} + \delta_{js}\right) \tag{38}$$
$$H_{2T} = \Gamma_{jT} \left(\delta_{jr}\psi_{rx} + \delta_{jx}\right) \tag{39}$$

Replace $j_{T+1} = j_T$ in equation 31 with equation 37.

$$r_T = \psi_{rs} s_T + \psi_{rj} \left(H_{1T} s_T + H_{2T} x_T\right) + \psi_{rx} x_T \tag{40}$$

Collecting terms, this can be expressed as a rule for r_T in terms of the state and exogenous variables in period T.

$$r_T = M_{1T} s_T + M_{2T} x_T \tag{41}$$

The coefficient matrices in equation 41 are defined as follows.

$$M_{1T} = \psi_{rsT} + \psi_{rj} H_{1T} \tag{42}$$
$$M_{2T} = \psi_{rxT} + \psi_{rj} H_{2T} \tag{43}$$

4.2 Further iterations for the stable manifold

Next, derive for the rule for period t assuming a specific form of the rule in period $t+1$ that includes the rules described in equations 37 and 41 as special cases for the terminal period. This iterative process, continues until the rule is far enough before the terminal period T that the coefficient matrices in the rule for the costate variable do not change with each iteration.

The rules for any time period t from $t=0$ to $t=T$ involve linear functions of future exogenous variables. For brevity, define these functions as follows:

(44) $$F_{jt} = f_{jt}(x_{t+1}, \ldots, x_T)$$

(45) $$F_{rt} = f_{rt}(x_{t+1}, \ldots, x_T)$$

(46) $$F_{st} = f_{st}(x_{t+1}, \ldots, x_T)$$

If the rules for period t have the same form as the rules for period $t+1$, a process of induction, back from period T, implies the rules for all time periods from period 0 through to period T and allows calculation of the coefficient matrices in those rules.

The assumed rules for period $t+1$ are set out below.

(47) $$j_{t+1} = H_{1t+1}\; s_{t+1} + H_{2t+1}\; x_{t+1} + F_{jt+1}$$

(48) $$r_{t+1} = M_{1t+1}\; s_{t+1} + M_{2t+1}\; x_{t+1} + F_{rt+1}$$

We need to check that the assumed rules in equations 47 and 48 imply the rules for j_t and r_t in equations 49 and 50.

(49) $$j_t = H_{1t}\; s_t + H_{2t}\; x_t + F_{jt}$$

(50) $$r_t = M_{1t}\; s_t + M_{2t}\; x_t + F_{rt}$$

Along with the SSF, the rules in equations 47 and 48 imply a rule for s_{t+1}. To show this implication, use the SSF equation for the state variables, equation 26, substituting r_{t+1} with the assumed rule for r_{t+1} in equation 48.

(51) $$s_{t+1} = \delta_{sr}\left(M_{1t+1}s_{t+1} + M_{2t+1}x_{t+1} + F_{rt+1}\right) + \delta_{ss}s_t + \delta_{sj}j_t + \delta_{sx}x_t$$

Gather all s_{t+1} terms on the left.

(52) $$\left(I_s - \delta_{sr}M_{1t+1}\right)s_{t+1} = \delta_{ss}s_t + \delta_{sj}j_t + \delta_{sx}x_t + \delta_{sr}\left(M_{2t+1}x_{t+1} + F_{rt+1}\right)$$

Multiply through by $\Gamma_{st} = \left(I_s - \delta_{sr}M_{1t+1}\right)^{-1}$ to obtain the rule for s_{t+1}.

(53) $$s_{t+1} = \tau_{sst}s_t + \tau_{sjt}j_t + \tau_{sxt}x_t + F_{st}$$

The coefficient matrices in equation 53 are defined as follows.

(54) $$\tau_{sst} = \Gamma_{st}\;\delta_{ss}$$

(55) $$\tau_{sjt} = \Gamma_{st}\;\delta_{sj}$$

(56) $$\tau_{sxt} = \Gamma_{st}\;\delta_{sx}$$

The linear function of future exogenous variables is defined as follows:

$$F_{st} = \Gamma_{st} \delta_{sr} \left(M_{2t+1} x_{t+1} + F_{rt+1} \right) \tag{57}$$

Using the three rules in equations 47, 48, and 53, obtain the coefficient matrices in the rules for j_t, r_t and s_t. Begin by equating the right-hand-sides of the assumed j_{t+1} rule, equation 47, and the SSF for j_{t+1}, equation 27.

$$H_{1t+1} s_{t+1} + H_{2t+1} x_{t+1} + F_{jt+1} = \delta_{jr} r_{t+1} + \delta_{js} s_t + \delta_{jj} j_t + \delta_{jx} x_t \tag{58}$$

Substitute r_{t+1} with the assumed rule for r_{t+1} in equation 48.

$$\begin{aligned} &H_{1t+1} s_{t+1} + H_{2t+1} x_{t+1} + F_{jt+1} \\ &= \delta_{jr} \left(M_{1t+1} s_{t+1} + M_{2t+1} x_{t+1} + F_{rt+1} \right) + \delta_{js} s_t + \delta_{jj} j_t + \delta_{jx} x_t \end{aligned} \tag{59}$$

Gather s_{t+1} terms.

$$\begin{aligned} &\left(H_{1t+1} - \delta_{jr} M_{1t+1} \right) s_{t+1} + H_{2t+1} x_{t+1} + F_{jt+1} \\ &= \delta_{js} s_t + \delta_{jj} j_t + \delta_{jx} x_t + \delta_{jr} \left(M_{2t+1} x_{t+1} + F_{rt+1} \right) \end{aligned} \tag{60}$$

Use the rule for s_{t+1} in equation 53 to eliminate s_{t+1}.

$$\begin{aligned} &\left(H_{1t+1} - \delta_{jr} M_{1t+1} \right) \left(\tau_{sst} s_t + \tau_{sjt} j_t + \tau_{sxt} x_t + F_{st} \right) + H_{2t+1} x_{t+1} + F_{jt+1} \\ &= \delta_{js} s_t + \delta_{jj} j_t + \delta_{jx} x_t + \delta_{jr} \left(M_{2t+1} x_{t+1} + F_{rt+1} \right) \end{aligned} \tag{61}$$

Rearrange to have only terms involving j_t on the left and gather like terms on the right.

$$\begin{aligned} &\left(\delta_{jj} - \left(H_{1t+1} - \delta_{jr} M_{1t+1} \right) \tau_{sjt} \right) j_t \\ &= \left(\left(H_{1t+1} - \delta_{jr} M_{1t+1} \right) \tau_{sst} - \delta_{js} \right) s_t \\ &+ \left(\left(H_{1t+1} - \delta_{jr} M_{1t+1} \right) \tau_{sxt} - \delta_{jx} \right) x_t \\ &+ H_{2t+1} x_{t+1} + F_{jt+1} + \left(H_{1t+1} - \delta_{jr} M_{1t+1} \right) F_{st} - \delta_{jr} \left(M_{2t+1} x_{t+1} + F_{rt+1} \right) \end{aligned} \tag{62}$$

Multiply through by $\Gamma_{jt} = \left(\delta_{jj} - \left(H_{1t+1} - \delta_{jr} M_{1t+1} \right) \tau_{sjt} \right)^{-1}$ to obtain the rule for j_t in equation 49.

The implied coefficient matrix definitions are:

$$H_{1t} = \Gamma_{jt} \left(\left(H_{1t+1} - \delta_{jr} M_{1t+1} \right) \tau_{sst} - \delta_{js} \right) \tag{63}$$

$$H_{2t} = \Gamma_{jt} \left(\left(H_{1t+1} - \delta_{jr} M_{1t+1} \right) \tau_{sxt} - \delta_{jx} \right) \tag{64}$$

The implied function of future exogenous variables in the rule for j_t is:

$$F_{jt} = \Gamma_{jt} \left(H_{2t+1} x_{t+1} + F_{jt+1} + \left(H_{1t+1} - \delta_{jr} M_{1t+1} \right) F_{st} - \delta_{jr} \left(M_{2t+1} x_{t+1} + F_{rt+1} \right) \right) \tag{65}$$

Next derive the rule for r_t in equation 50.

To derive the rule for r_t, replace r_{t+1} with the assumed rule for r_{t+1} in the SSF equation for r_t, equation 28.

$$r_t = \delta_{rr}\left(M_{1t+1}s_{t+1} + M_{2t+1}x_{t+1} + F_{rt+1}\right) + \delta_{rs}s_t + \delta_{rj}j_t + \delta_{rx}x_t \tag{66}$$

Substitute for s_{t+1} with the rule in equation 53.

$$r_t = \delta_{rr}\left(M_{1t+1}\left(\tau_{sst}s_t + \tau_{sjt}j_t + \tau_{sxt}x_t + F_{st}\right) + M_{2t+1}x_{t+1} + F_{rt+1}\right) + \delta_{rs}s_t + \delta_{rj}j_t + \delta_{rx}x_t \tag{67}$$

Gather like terms on the right.

$$\begin{aligned} r_t = &\left(\delta_{rs} + \delta_{rr}M_{1t+1}\tau_{sst}\right)s_t \\ &+ \left(\delta_{rj} + \delta_{rr}M_{1t+1}\tau_{sjt}\right)j_t \\ &+ \left(\delta_{rx} + \delta_{rr}M_{1t+1}\tau_{sxt}\right)x_t \\ &+ \delta_{rr}\left(M_{1t+1}F_{st} + M_{2t+1}x_{t+1} + F_{rt+1}\right) \end{aligned} \tag{68}$$

Replace j_t with its rule in equation 49 to obtain the required form for the rule for r_t.

$$\begin{aligned} r_t = &\left(\delta_{rs} + \delta_{rr}M_{1t+1}\tau_{sst}\right)s_t \\ &+ \left(\delta_{rj} + \delta_{rr}M_{1t+1}\tau_{sjt}\right)\left(H_{1t}s_t + H_{2t}x_t + F_{jt}\right) \\ &+ \left(\delta_{rx} + \delta_{rr}M_{1t+1}\tau_{sxt}\right)x_t \\ &+ \delta_{rr}\left(M_{1t+1}F_{st} + M_{2t+1}x_{t+1} + F_{rt+1}\right) \end{aligned} \tag{69}$$

Gathering like terms on the right produces the rule for r_t in equation 50 as required.

$$\begin{aligned} r_t = &\left(\delta_{rs} + \delta_{rr}M_{1t+1}\tau_{sst} + \left(\delta_{rj} + \delta_{rr}M_{1t+1}\tau_{sjt}\right)H_{1t}\right)s_t \\ &\left(\delta_{rx} + \delta_{rr}M_{1t+1}\tau_{sxt} + \left(\delta_{rj} + \delta_{rr}M_{1t+1}\tau_{sjt}\right)H_{2t}\right)x_t \\ &+ \delta_{rr}\left(M_{1t+1}F_{st} + M_{2t+1}x_{t+1} + F_{rt+1}\right) + \left(\delta_{rj} + \delta_{rr}M_{1t+1}\tau_{sjt}\right)F_{jt} \end{aligned} \tag{70}$$

The implied coefficient matrix definitions are:

$$M_{1t} = \delta_{rs} + \delta_{rr}M_{1t+1}\left(\tau_{sst} + \tau_{sjt}H_{1t}\right) + \delta_{rj}H_{1t} \tag{71}$$

$$M_{2t} = \delta_{rx} + \delta_{rr}M_{1t+1}\left(\tau_{sxt} + \tau_{sjt}H_{2t}\right) + \delta_{rj}H_{2t} \tag{72}$$

The implied function of future exogenous variables in the rule for s_t is:

$$F_{rt} = \delta_{rr}\left(M_{1t+1}F_{st} + M_{2t+1}x_{t+1} + F_{rt+1}\right) + \left(\delta_{rj} + \delta_{rr}M_{1t+1}\tau_{sjt}\right)F_{jt} \tag{73}$$

This result confirms that the rules for r_t and j_t, have the form specified in 50 and 49 assuming the form for the rules for r_{t+1} and j_{t+1} in equations 47 and 48. Given that the rules for j_T and r_T are special cases of the form specified in equations 47 and 48, we have proved that the rules have the form in equations 50 and 49, for all time periods $t = 0$ through to $t = T$.

We have also derived the relationship between the coefficient matrices in the rules for all time periods and have derived the rules for the state variables in all time periods and the relationship between the coefficient matrices in those rules and the coefficient matrices in the SSF and the rules for j and r.

4.2.1 A rule for ${}_tr_{t+1}$ in terms of s_t

A rule for ${}_tr_{t+1} = r_{t+1}$ in terms of s_t and exogenous variables is helpful when generating model projections. It also underpins an alternative way of deriving the rule for r_t in terms of s_t and exogenous variables.

To obtain the rule expressing r_{t+1} in terms of s_t and exogenous variables, replace s_{t+1} in equation 48 with the right side of equation 53.

$$r_{t+1} = M_{1t+1}\left(\tau_{sst}s_t + \tau_{sjt}j_t + \tau_{sxt}x_t + F_{st}\right) + M_{2t+1}\, x_{t+1} + F_{rt+1} \tag{74}$$

Eliminate j_t using equation 49.

$$r_{t+1} = M_{1t+1}\left(\tau_{sst}s_t + \tau_{sjt}\left(H_{1t}\, s_t + H_{2t}\, x_t + F_{jt}\right) + \tau_{sxt}x_t + F_{st}\right) + M_{2t+1}\, x_{t+1} + F_{rt+1} \tag{75}$$

Collect like terms on the right.

$$\begin{aligned} r_{t+1} = {} & M_{1t+1}\left(\tau_{sst} + \tau_{sjt}H_{1t}\right)s_t \\ & + M_{1t+1}\left(\tau_{sxt} + \tau_{sjt}H_{2t}\right)x_t \\ & + M_{1t+1}\left(F_{st} + \tau_{sjt}F_{jt}\right) + M_{2t+1}x_{t+1} + F_{rt+1} \end{aligned} \tag{76}$$

Define the following coefficient matrices.

$$\mu_{1t} = M_{1t+1}\left(\tau_{sst} + \tau_{sjt}H_{1t}\right) \tag{77}$$

$$\mu_{2t} = M_{1t+1}\left(\tau_{sxt} + \tau_{sjt}H_{2t}\right) \tag{78}$$

Define the following function of time t and future exogenous variables.

$$G_{rt} = M_{1t+1}\left(F_{st} + \tau_{sjt}F_{jt}\right) + M_{2t+1}x_{t+1} + F_{rt+1} \tag{79}$$

Using these definitions, we can express a the rule for r_{t+1} in terms of s_t and x_t and future exogenous variables.

$$r_{t+1} = \mu_{1t}s_t + \mu_{2t}x_t + G_{rt} \tag{80}$$

The rule for r_t then obtains by eliminating ${}_tr_{t+1}$ and j_t from the SSF equation for r_t, equation 28, using and use equations equation 80 and 49 respectively.

$$r_t = \delta_{rr}\left(\mu_{1t}s_t + \mu_{2t}x_t + G_{rt}\right) + \delta_{rs}s_t + \delta_{rj}\left(H_{1t}s_t + H_{2t}x_t + F_{jt}\right) + \delta_{rx}x_t \tag{81}$$

Collect like terms on the right to obtain the required rule for r_t.

$$r_t = \left(\delta_{rs} + \delta_{rr}\mu_{1t} + \delta_{rj}H_{1t}\right)s_t + \left(\delta_{rx} + \delta_{rr}\mu_{2t} + \delta_{rj}H_{2t}\right)x_t + \delta_{rr}G_{rt} + \delta_{rj}F_{jt} \tag{82}$$

The implied coefficient matrix definitions are:

$$M_{1t} = \left(\delta_{rs} + \delta_{rr}\mu_{1t} + \delta_{rj}H_{1t}\right) \tag{83}$$

$$M_{2t} = \left(\delta_{rx} + \delta_{rr}\mu_{2t} + \delta_{rj}H_{2t}\right) \tag{84}$$

It can be shown that these definitions are equivalent to the definitions in equations 71 and 72.

The implied function of future exogenous variables in the rule for s_t is:

$$F_{rt} = \delta_{rr}G_{rt} + \delta_{rj}F_{jt} \tag{85}$$

This definition is also equivalent to that in equation 73.

4.3 Convergence of the stable manifold

Convergence is deemed to have happened when the matrices, H_{1t}, H_{2t}, M_{1t} and M_{2t} stabilise to be independent of the time period, H_1, H_2, M_1 and M_2.

As shown by Blanchard and Kahn (1980), the stable manifold is only unique if *matrix*? has exactly one eigenvalue outside the unit circle for each costate variable in the model. Otherwise the solution is not unique. It would be helpful to be explicit about how the linearised model in this paper maps to the linear model in 1a of Blanchard and Kahn (1980) so that we can determine how to assess uniqueness based on the J matrix (J is from the Jordan canonical form of the full linear model coefficient matrix) eigenvalues when we form the linear model.

What can we say about whether convergence is guaranteed? Is there a connection between uniqueness and a guarantee of convergence?

4.4 State dynamics and system stability

To check that the stable manifold does imply stability of the system, check the eigenvalues of the matrix in the rule relating the state in period $t+1$ to the state in period t, obtained after substituting the rules j_t and r_{t+1} back into the SSF equation for the state variables. All of those eigenvalues must lie inside the unit circle if the system is stable.

To implement this stability test and to obtain a rule for the state dynamics, it is useful to write the rule for the state variables as a function of state variables in the previous period and exogenous variables.

To do so, substitute the rule for j_t, equation 49, and the rule for r_{t+1}, equation 80, into the SSF equation for the state variables, equation 26 where t is a sufficient number of periods before T for the rules to be expressed in terms of H_1, H_2, M_1 and M_2.

$$s_{t+1} = \delta_{sr}\left(\mu_{1t}s_t + \mu_{2t}x_t + G_{rt}\right) + \delta_{ss}s_t + \delta_{sj}\left(H_1 s_t + H_2 x_t + F_{jt}\right) + \delta_{sx}x_t \tag{86}$$

Collect like terms on the right.

$$s_{t+1} = \left(\delta_{ss} + \delta_{sr}\mu_{1t} + \delta_{sj}H_1\right)s_t + \left(\delta_{sx} + \delta_{sr}\mu_{2t} + \delta_{sj}H_2\right)x_t + \delta_{sr}G_{rt} + \delta_{sj}F_{jt} \tag{87}$$

Define the following coefficient matrices, N_1 and N_2.

$$N_1 = \Delta\left(\delta_{ss} + \delta_{sj}H_1\right) \tag{88}$$
$$N_2 = \Delta\left(\delta_{sx} + \delta_{sj}H_2\right) \tag{89}$$

Define the function of future exogenous variables G_{st}:

$$G_{st} = \Delta\left(\delta_{sr}\left(M_2 x_{t+1} + F_{rt+1}\right) + \delta_{sj}F_{jt}\right) \tag{90}$$

The state transition equation can then be written as:

$$s_{t+1} = N_1 s_t + N_2 x_t + G_{st} \tag{91}$$

The matrix to consider when assessing stability is N_1, the state transition matrix. Stability requires all eigenvalues of this matrix to be inside the unit circle.

References

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