

Solving G-Cubed Models without policy optimisation

Warwick J McKibbin and Geoffrey Shuetrim

McKibbin Software Group

May 19, 2023

Abstract

This paper sets out the algebra involved in solving the G-Cubed class of macroeconomic models, without allowing for optimal policy. Policy responses are instead captured by exogenously determined policy response functions. It provides the line-by-line algebraic details needed to implement the two-point boundary solution algorithm described in ? that builds on the work of ?.

1 The non-linear model

The model can be expressed in terms of the following variables:

- the vector of state variables, S_t ;
- the vector of costate or “jump” variables, J_t ;
- the vector, R_t , the subset of endogenous variables that also enter the model in expectations, at time, t of their value in time $t + 1$, $E_t(R_{t+1})$;
- the vector of other endogenous variables, Z_t ; and
- the vector of exogenous variables, X_t .

In non-linear form, these equations can be written:

$$(1) \quad S_{t+1} = \Phi_S(S_{t+1}, E_t(J_{t+1}), R_t, Z_t, E_t(R_{t+1}), S_t, J_t, X_t)$$

$$(2) \quad E_t(J_{t+1}) = \Phi_J(S_{t+1}, E_t(J_{t+1}), R_t, Z_t, E_t(R_{t+1}), S_t, J_t, X_t)$$

$$(3) \quad R_t = \Phi_R(S_{t+1}, E_t(J_{t+1}), R_t, Z_t, E_t(R_{t+1}), S_t, J_t, X_t)$$

$$(4) \quad Z_t = \Phi_Z(S_{t+1}, E_t(J_{t+1}), R_t, Z_t, E_t(R_{t+1}), S_t, J_t, X_t)$$

2 The linear model as a series of differential equations

The model is linearised using a first-order Taylor-series expansion around a point P :

$$P = (\bar{S}_{t+1}, \bar{S}_t, E_t(\bar{J}_{t+1}), \bar{J}_t, \bar{R}_t, E_t(\bar{R}_{t+1}), \bar{Z}_t, \bar{X}_t)$$

In most cases, P is defined as values for the variables in the year prior to the start of model projections.

Adjustments are made to the values selected for variables that are a long way from their equilibrium values, for example, when real interest rates are negative, those negative interest rates will not be used as part of the point around which the model is linearised.

Calculating numerical derivatives at the linearisation point, the model can be approximated by a set of linear differential equations. For A in (S, J, R, Z) , define ϕ_{An} as the matrix of partial derivatives of function $\Phi_A()$ with respect to its n th argument.

To improve readability, define the following notation in time-period t :

$$(5) \quad \begin{aligned} dS_t &= s_t \\ dJ_t &= j_t \\ dR_t &= r_t \\ dZ_t &= z_t \\ dX_t &= x_t \\ E_t(r_{t+1}) &= {}_t r_{t+1} \\ E_t(j_{t+1}) &= {}_t j_{t+1} \end{aligned}$$

The linear model is then the following differential equations.

$$(6) \quad s_{t+1} = \phi_{s1} s_{t+1} + \phi_{s2} {}_t j_{t+1} + \phi_{s3} r_t + \phi_{s4} z_t + \phi_{s5} {}_t r_{t+1} + \phi_{s6} s_t + \phi_{s7} j_t + \phi_{s8} x_t$$

$$(7) \quad {}_t j_{t+1} = \phi_{j1} s_{t+1} + \phi_{j2} {}_t j_{t+1} + \phi_{j3} r_t + \phi_{j4} z_t + \phi_{j5} {}_t r_{t+1} + \phi_{j6} s_t + \phi_{j7} j_t + \phi_{j8} x_t$$

$$(8) \quad r_t = \phi_{r1} s_{t+1} + \phi_{r2} {}_t j_{t+1} + \phi_{r3} r_t + \phi_{r4} z_t + \phi_{r5} {}_t r_{t+1} + \phi_{r6} s_t + \phi_{r7} j_t + \phi_{r8} x_t$$

$$(9) \quad z_t = \phi_{z1} s_{t+1} + \phi_{z2} {}_t j_{t+1} + \phi_{z3} r_t + \phi_{z4} z_t + \phi_{z5} {}_t r_{t+1} + \phi_{z6} s_t + \phi_{z7} j_t + \phi_{z8} x_t$$

3 The State-Space Form

The State-Space Form (SSF) is obtained by eliminating s_{t+1} , ${}_t j_{t+1}$, r_t , and z_t from the right-hand side of the linear model.

Working through the necessary steps, first rearrange equation ?? to gather all z_t terms on the left:

$$(10) \quad (I_z - \phi_{z4})z_t = \phi_{z1} s_{t+1} + \phi_{z2} {}_t j_{t+1} + \phi_{z3} r_t + \phi_{z5} {}_t r_{t+1} + \phi_{z6} s_t + \phi_{z7} j_t + \phi_{z8} x_t$$

Multiplying through by the inverse of $(I_z - \phi_{z4})$ and defining the α coefficient matrices appropriately, we obtain:

$$(11) \quad z_t = \alpha_{z1} s_{t+1} + \alpha_{z2} {}_t j_{t+1} + \alpha_{z3} r_t + \alpha_{z5} {}_t r_{t+1} + \alpha_{z6} s_t + \alpha_{z7} j_t + \alpha_{z8} x_t$$

Use equation ??, to substitute z_t out of equations ??, ??, and ??, again defining α coefficient matrices appropriately:

$$(12) \quad s_{t+1} = \alpha_{s1}s_{t+1} + \alpha_{s2} \, {}_tj_{t+1} + \alpha_{s3}r_t + \alpha_{s5} \, {}_tr_{t+1} + \alpha_{s6}s_t + \alpha_{s7}j_t + \alpha_{s8}x_t$$

$$(13) \quad {}_tj_{t+1} = \alpha_{j1}s_{t+1} + \alpha_{j2} \, {}_tj_{t+1} + \alpha_{j3}r_t + \alpha_{j5} \, {}_tr_{t+1} + \alpha_{j6}s_t + \alpha_{j7}j_t + \alpha_{j8}x_t$$

$$(14) \quad r_t = \alpha_{r1}s_{t+1} + \alpha_{r2} \, {}_tj_{t+1} + \alpha_{r3}r_t + \alpha_{r5} \, {}_tr_{t+1} + \alpha_{r6}s_t + \alpha_{r7}j_t + \alpha_{r8}x_t$$

Repeat the process to eliminate r_t from the right of the linear model, rearranging equation ??:

$$(15) \quad (I_r - \alpha_{r3})r_t = \alpha_{r1}s_{t+1} + \alpha_{r2} \, {}_tj_{t+1} + \alpha_{r4}z_t + \alpha_{r5} \, {}_tr_{t+1} + \alpha_{r6}s_t + \alpha_{r7}j_t + \alpha_{r8}x_t$$

Multiply through by the inverse of $(I_r - \alpha_{r3})$, to solve for r_t , defining the β coefficient matrices appropriately.

$$(16) \quad r_t = \beta_{r1}s_{t+1} + \beta_{r2} \, {}_tj_{t+1} + \beta_{r5} \, {}_tr_{t+1} + \beta_{r6}s_t + \beta_{r7}j_t + \beta_{r8}x_t$$

Use equation ??, to substitute r_t out of equations ??, ??, and ??, defining the β coefficient matrices appropriately.

$$(17) \quad s_{t+1} = \beta_{s1}s_{t+1} + \beta_{s2} \, {}_tj_{t+1} + \beta_{s5} \, {}_tr_{t+1} + \beta_{s6}s_t + \beta_{s7}j_t + \beta_{s8}x_t$$

$$(18) \quad {}_tj_{t+1} = \beta_{j1}s_{t+1} + \beta_{j2} \, {}_tj_{t+1} + \beta_{j5} \, {}_tr_{t+1} + \beta_{j6}s_t + \beta_{j7}j_t + \beta_{j8}x_t$$

$$(19) \quad z_t = \beta_{z1}s_{t+1} + \beta_{z2} \, {}_tj_{t+1} + \beta_{z5} \, {}_tr_{t+1} + \beta_{z6}s_t + \beta_{z7}j_t + \beta_{z8}x_t$$

Repeat the process to eliminate ${}_tj_{t+1}$ from the right of the linear model, rearranging equation ??:

$$(20) \quad (I_j - \beta_{j2}) \, {}_tj_{t+1} = \beta_{j1}s_{t+1} + \beta_{j5} \, {}_tr_{t+1} + \beta_{j6}s_t + \beta_{j7}j_t + \beta_{j8}x_t$$

Multiply through by the inverse of $(I_j - \beta_{j2})$ to solve for ${}_tj_{t+1}$, defining the γ coefficient matrices appropriately.

$$(21) \quad {}_t\dot{j}_{t+1} = \gamma_{j1}s_{t+1} + \gamma_{j5} {}_t r_{t+1} + \gamma_{j6}s_{t-1} + \gamma_{j7}\dot{j}_t + \gamma_{j8}x_t$$

Use equation ?? to substitute ${}_t\dot{j}_{t+1}$ out of equations ??, ??, and ??, defining the γ coefficient matrices appropriately:

$$(22) \quad s_{t+1} = \gamma_{s1}s_{t+1} + \gamma_{s5} {}_t r_{t+1} + \gamma_{s6}s_t + \gamma_{s7}\dot{j}_t + \gamma_{s8}x_t$$

$$(23) \quad r_t = \gamma_{r1}s_{t+1} + \gamma_{r5} {}_t r_{t+1} + \gamma_{r6}s_t + \gamma_{r7}\dot{j}_t + \gamma_{r8}x_t$$

$$(24) \quad z_t = \gamma_{z1}s_{t+1} + \gamma_{z5} {}_t r_{t+1} + \gamma_{z6}s_t + \gamma_{z7}\dot{j}_t + \gamma_{z8}x_t$$

Repeat the process to eliminate s_{t+1} from the right of the linear model, rearranging equation ??:

$$(25) \quad (I_s - \gamma_{s1})s_{t+1} = \gamma_{s5} {}_t r_{t+1} + \gamma_{s6}s_t + \gamma_{s7}\dot{j}_t + \gamma_{s8}x_t$$

Multiply through by the inverse of $(I_s - \gamma_{s1})$, to solve for s_{t+1} , defining the δ coefficient matrices appropriately.

$$(26) \quad s_{t+1} = \delta_{sr} {}_t r_{t+1} + \delta_{ss} s_t + \delta_{sj} j_t + \delta_{sx} x_t$$

Use equation ?? to substitute s_{t+1} out of equations ??, ??, and ??, defining the δ coefficient matrices appropriately. Note that the δ coefficient matrices are referenced throughout the rest of this document. Their subscripts indicate the vectors that they relate, with the first subscript indicating the vector on the left of the equation and the second subscript indicating the vector on the right of the equation. Thus, δ_{sj} is the matrix of SSF coefficients describing the relationship between s_{t+1} on the left and j_t on the right.

$$(27) \quad {}_t j_{t+1} = \delta_{jr} {}_t r_{t+1} + \delta_{js} s_t + \delta_{jj} j_t + \delta_{jx} x_t$$

$$(28) \quad r_t = \delta_{rr} {}_t r_{t+1} + \delta_{rs} s_t + \delta_{rj} j_t + \delta_{rx} x_t$$

$$(29) \quad z_t = \delta_{zr} {}_t r_{t+1} + \delta_{zs} s_t + \delta_{zj} j_t + \delta_{zx} x_t$$

Equations ??, ??, ??, and ?? constitute the SSF.

4 The stable manifold

The stable manifold is the non-exploding path for the costate variables, j_t . It is found using the iterative algorithm described in ?.

4.1 The first iteration to find the stable manifold

The model needs to be solved for paths for all variables from period $t = 0$ to period $t = T$. Use a suitable terminal condition to start the iterative process of computing the transition rule for the costate variable that implies model consistent expectations.

With (linearised) model-consistent expectations, we can slightly simplify notation again, replacing ${}_t r_{t+1}$ and ${}_t j_{t+1}$ with r_{t+1} and j_{t+1} respectively because the model has no sources of uncertainty.

The first iteration yields rules for j_T , r_T in terms of s_T and x_T . These rules must be model consistent with remaining periods. Being the final period, T , a terminal condition is required.

An intuitive terminal condition involves the period-to-period change in the variables being zero from period T to period $T + 1$ so $j_T = j_{T+1}$ and $r_T = r_{T+1}$. This enables elimination of r_t and j_t from the period T SSF equations.

First eliminate $r_{T+1} = r_T$ from the right of equation ??.

$$(30) \quad (I_r - \delta_{rr}) r_T = \delta_{rs} s_T + \delta_{rj} j_T + \delta_{rx} x_T$$

Multiply through by $\Gamma_{rT} = (I_r - \delta_{rr})^{-1}$.

$$(31) \quad r_T = \psi_{rs}s_T + \psi_{rj}j_T + \psi_{rx}x_T$$

The coefficient matrices in equation ?? are defined as follows.

$$(32) \quad \psi_{rs} = \Gamma_{rT}\delta_{rs}$$

$$(33) \quad \psi_{rj} = \Gamma_{rT}\delta_{rj}$$

$$(34) \quad \psi_{rx} = \Gamma_{rT}\delta_{rx}$$

Using the terminal condition again, $r_{T+1} = r_T$ substitute equation ?? for r_{T+1} in equation ??.

$$(35) \quad \begin{aligned} j_{T+1} &= \delta_{jr}(\psi_{rs}s_T + \psi_{rj}j_T + \psi_{rx}x_T) \\ &+ \delta_{js}s_T + \delta_{jj}j_T + \delta_{jx}x_T \end{aligned}$$

Given the terminal condition implies $j_{T+1} = j_T$, collect j_T terms on the left.

$$(36) \quad \begin{aligned} (I_j - \delta_{jj} - \delta_{jr}\psi_{rj})j_T &= \delta_{jr}(\psi_{rs}s_T + \psi_{rx}x_T) \\ &+ \delta_{js}s_T + \delta_{jx}x_T \end{aligned}$$

Multiply through by $\Gamma_{jT} = (I_j - \delta_{jj} - \delta_{jr}\psi_{rj})^{-1}$ to obtain a rule for j_T in terms of s_T and x_T .

$$(37) \quad j_T = H_{1T}s_T + H_{2T}x_T$$

The coefficient matrices in equation ?? are defined as follows.

$$(38) \quad H_{1T} = \Gamma_{jT}(\delta_{jr}\psi_{rs} + \delta_{js})$$

$$(39) \quad H_{2T} = \Gamma_{jT}(\delta_{jr}\psi_{rx} + \delta_{jx})$$

Replace $j_{T+1} = j_T$ in equation ?? with equation ??.

$$(40) \quad r_T = \psi_{rs}s_T + \psi_{rj}(H_{1T}s_T + H_{2T}x_T) + \psi_{rx}x_T$$

Collecting terms, this can be expressed as a rule for r_T in terms of the state and exogenous variables in period T .

$$(41) \quad r_T = M_{1T}s_T + M_{2T}x_T$$

The coefficient matrices in equation ?? are defined as follows.

$$(42) \quad M_{1T} = \psi_{rsT} + \psi_{rj}H_{1T}$$

$$(43) \quad M_{2T} = \psi_{rxT} + \psi_{rj}H_{2T}$$

4.2 Further iterations for the stable manifold

Next, derive for the rule for period t assuming a specific form of the rule in period $t + 1$ that includes the rules described in equations ?? and ?? as special cases for the terminal period. This iterative process, continues until the rule is far enough before the terminal period T that the coefficient matrices in the rule for the costate variable do not change with each iteration.

The rules for any time period t from $t = 0$ to $t = T$ involve linear functions of future exogenous variables. For brevity, define these functions as follows:

$$(44) \quad F_{jt} = f_{jt}(x_{t+1}, \dots, x_T)$$

$$(45) \quad F_{rt} = f_{rt}(x_{t+1}, \dots, x_T)$$

$$(46) \quad F_{st} = f_{st}(x_{t+1}, \dots, x_T)$$

If the rules for period t have the same form as the rules for period $t + 1$, a process of induction, back from period T , implies the rules for all time periods from period 0 through to period T and allows calculation of the coefficient matrices in those rules.

The assumed rules for period $t + 1$ are set out below.

$$(47) \quad j_{t+1} = H_{1t+1} s_{t+1} + H_{2t+1} x_{t+1} + F_{jt+1}$$

$$(48) \quad r_{t+1} = M_{1t+1} s_{t+1} + M_{2t+1} x_{t+1} + F_{rt+1}$$

We need to check that the assumed rules in equations ?? and ?? imply the rules for j_t and r_t in equations ?? and ??.

$$(49) \quad j_t = H_{1t} s_t + H_{2t} x_t + F_{jt}$$

$$(50) \quad r_t = M_{1t} s_t + M_{2t} x_t + F_{rt}$$

Along with the SSF, the rules in equations ?? and ?? imply a rule for s_{t+1} . To show this implication, use the SSF equation for the state variables, equation ??, substituting r_{t+1} with the assumed rule for r_{t+1} in equation ??.

$$(51) \quad s_{t+1} = \delta_{sr} (M_{1t+1} s_{t+1} + M_{2t+1} x_{t+1} + F_{rt+1}) + \delta_{ss} s_t + \delta_{sj} j_t + \delta_{sx} x_t$$

Gather all s_{t+1} terms on the left.

$$(52) \quad (I_s - \delta_{sr} M_{1t+1}) s_{t+1} = \delta_{ss} s_t + \delta_{sj} j_t + \delta_{sx} x_t + \delta_{sr} (M_{2t+1} x_{t+1} + F_{rt+1})$$

Multiply through by $\Gamma_{st} = (I_s - \delta_{sr} M_{1t+1})^{-1}$ to obtain the rule for s_{t+1} .

$$(53) \quad s_{t+1} = \tau_{sst} s_t + \tau_{sjt} j_t + \tau_{sxt} x_t + F_{st}$$

The coefficient matrices in equation ?? are defined as follows.

$$(54) \quad \tau_{sst} = \Gamma_{st} \delta_{ss}$$

$$(55) \quad \tau_{sjt} = \Gamma_{st} \delta_{sj}$$

$$(56) \quad \tau_{sxt} = \Gamma_{st} \delta_{sx}$$

The linear function of future exogenous variables is defined as follows:

$$(57) \quad F_{st} = \Gamma_{st} \delta_{sr} (M_{2t+1} x_{t+1} + F_{rt+1})$$

Using the three rules in equations ??, ??, and ??, obtain the coefficient matrices in the rules for j_t , r_t and s_t . Begin by equating the right-hand-sides of the assumed j_{t+1} rule, equation ??, and the SSF for j_{t+1} , equation ??.

$$(58) \quad H_{1t+1} s_{t+1} + H_{2t+1} x_{t+1} + F_{jt+1} = \delta_{jr} r_{t+1} + \delta_{js} s_t + \delta_{jj} j_t + \delta_{jx} x_t$$

Substitute r_{t+1} with the assumed rule for r_{t+1} in equation ??.

$$(59) \quad \begin{aligned} H_{1t+1} s_{t+1} + H_{2t+1} x_{t+1} + F_{jt+1} \\ = \delta_{jr} (M_{1t+1} s_{t+1} + M_{2t+1} x_{t+1} + F_{rt+1}) + \delta_{js} s_t + \delta_{jj} j_t + \delta_{jx} x_t \end{aligned}$$

Gather s_{t+1} terms.

$$(60) \quad \begin{aligned} (H_{1t+1} - \delta_{jr} M_{1t+1}) s_{t+1} + H_{2t+1} x_{t+1} + F_{jt+1} \\ = \delta_{js} s_t + \delta_{jj} j_t + \delta_{jx} x_t + \delta_{jr} (M_{2t+1} x_{t+1} + F_{rt+1}) \end{aligned}$$

Use the rule for s_{t+1} in equation ?? to eliminate s_{t+1} .

$$(61) \quad \begin{aligned} (H_{1t+1} - \delta_{jr} M_{1t+1}) (\tau_{sst} s_t + \tau_{sjt} j_t + \tau_{sxt} x_t + F_{st}) + H_{2t+1} x_{t+1} + F_{jt+1} \\ = \delta_{js} s_t + \delta_{jj} j_t + \delta_{jx} x_t + \delta_{jr} (M_{2t+1} x_{t+1} + F_{rt+1}) \end{aligned}$$

Rearrange to have only terms involving j_t on the left and gather like terms on the right.

$$(62) \quad \begin{aligned} (\delta_{jj} - (H_{1t+1} - \delta_{jr} M_{1t+1}) \tau_{sjt}) j_t \\ = ((H_{1t+1} - \delta_{jr} M_{1t+1}) \tau_{sst} - \delta_{js}) s_t \\ + ((H_{1t+1} - \delta_{jr} M_{1t+1}) \tau_{sxt} - \delta_{jx}) x_t \\ + H_{2t+1} x_{t+1} + F_{jt+1} + (H_{1t+1} - \delta_{jr} M_{1t+1}) F_{st} - \delta_{jr} (M_{2t+1} x_{t+1} + F_{rt+1}) \end{aligned}$$

Multiply through by $\Gamma_{jt} = (\delta_{jj} - (H_{1t+1} - \delta_{jr} M_{1t+1}) \tau_{sjt})^{-1}$ to obtain the rule for j_t in equation ??.

The implied coefficient matrix definitions are:

$$(63) \quad H_{1t} = \Gamma_{jt} ((H_{1t+1} - \delta_{jr} M_{1t+1}) \tau_{sst} - \delta_{js})$$

$$(64) \quad H_{2t} = \Gamma_{jt} ((H_{1t+1} - \delta_{jr} M_{1t+1}) \tau_{sxt} - \delta_{jx})$$

The implied function of future exogenous variables in the rule for j_t is:

$$(65) \quad F_{jt} = \Gamma_{jt} (H_{2t+1} x_{t+1} + F_{jt+1} + (H_{1t+1} - \delta_{jr} M_{1t+1}) F_{st} - \delta_{jr} (M_{2t+1} x_{t+1} + F_{rt+1}))$$

Next derive the rule for r_t in equation ??.

To derive the rule for r_t , replace r_{t+1} with the assumed rule for r_{t+1} in the SSF equation for r_t , equation ??.

$$(66) \quad r_t = \delta_{rr} (M_{1t+1}s_{t+1} + M_{2t+1}x_{t+1} + F_{rt+1}) + \delta_{rs}s_t + \delta_{rj}j_t + \delta_{rx}x_t$$

Substitute for s_{t+1} with the rule in equation ??.

$$(67) \quad r_t = \delta_{rr} (M_{1t+1}(\tau_{sst}s_t + \tau_{sjt}j_t + \tau_{sxt}x_t + F_{st}) + M_{2t+1}x_{t+1} + F_{rt+1}) + \delta_{rs}s_t + \delta_{rj}j_t + \delta_{rx}x_t$$

Gather like terms on the right.

$$(68) \quad \begin{aligned} r_t = & (\delta_{rs} + \delta_{rr}M_{1t+1}\tau_{sst}) s_t \\ & + (\delta_{rj} + \delta_{rr}M_{1t+1}\tau_{sjt}) j_t \\ & + (\delta_{rx} + \delta_{rr}M_{1t+1}\tau_{sxt}) x_t \\ & + \delta_{rr} (M_{1t+1}F_{st} + M_{2t+1}x_{t+1} + F_{rt+1}) \end{aligned}$$

Replace j_t with its rule in equation ?? to obtain the required form for the rule for r_t .

$$(69) \quad \begin{aligned} r_t = & (\delta_{rs} + \delta_{rr}M_{1t+1}\tau_{sst}) s_t \\ & + (\delta_{rj} + \delta_{rr}M_{1t+1}\tau_{sjt}) (H_{1t}s_t + H_{2t}x_t + F_{jt}) \\ & + (\delta_{rx} + \delta_{rr}M_{1t+1}\tau_{sxt}) x_t \\ & + \delta_{rr} (M_{1t+1}F_{st} + M_{2t+1}x_{t+1} + F_{rt+1}) \end{aligned}$$

Gathering like terms on the right produces the rule for r_t in equation ?? as required.

$$(70) \quad \begin{aligned} r_t = & (\delta_{rs} + \delta_{rr}M_{1t+1}\tau_{sst} + (\delta_{rj} + \delta_{rr}M_{1t+1}\tau_{sjt}) H_{1t}) s_t \\ & (\delta_{rx} + \delta_{rr}M_{1t+1}\tau_{sxt} + (\delta_{rj} + \delta_{rr}M_{1t+1}\tau_{sjt}) H_{2t}) x_t \\ & + \delta_{rr} (M_{1t+1}F_{st} + M_{2t+1}x_{t+1} + F_{rt+1}) + (\delta_{rj} + \delta_{rr}M_{1t+1}\tau_{sjt}) F_{jt} \end{aligned}$$

The implied coefficient matrix definitions are:

$$(71) \quad M_{1t} = \delta_{rs} + \delta_{rr}M_{1t+1}(\tau_{sst} + \tau_{sjt}H_{1t}) + \delta_{rj}H_{1t}$$

$$(72) \quad M_{2t} = \delta_{rx} + \delta_{rr}M_{1t+1}(\tau_{sxt} + \tau_{sjt}H_{2t}) + \delta_{rj}H_{2t}$$

The implied function of future exogenous variables in the rule for s_t is:

$$(73) \quad F_{rt} = \delta_{rr} (M_{1t+1}F_{st} + M_{2t+1}x_{t+1} + F_{rt+1}) + (\delta_{rj} + \delta_{rr}M_{1t+1}\tau_{sjt}) F_{jt}$$

This result confirms that the rules for r_t and j_t , have the form specified in ?? and ?? assuming the form for the rules for r_{t+1} and j_{t+1} in equations ?? and ??. Given that the rules for j_T and r_T are special cases of the form specified in equations ?? and ??, we have proved that the rules have the form in equations ?? and ??, for all time periods $t = 0$ through to $t = T$.

We have also derived the relationship between the coefficient matrices in the rules for all time periods and have derived the rules for the state variables in all time periods and the relationship between the coefficient matrices in those rules and the coefficient matrices in the SSF and the rules for j and r .

4.2.1 A rule for r_{t+1} in terms of s_t

A rule for r_{t+1} in terms of s_t and exogenous variables is helpful when generating model projections. It also underpins an alternative way of deriving the rule for r_t in terms of s_t and exogenous variables.

To obtain the rule expressing r_{t+1} in terms of s_t and exogenous variables, replace s_{t+1} in equation ?? with the right side of equation ??.

$$(74) \quad r_{t+1} = M_{1t+1} (\tau_{sst}s_t + \tau_{sjt}j_t + \tau_{sxt}x_t + F_{st}) + M_{2t+1} x_{t+1} + F_{rt+1}$$

Eliminate j_t using equation ??.

$$(75) \quad r_{t+1} = M_{1t+1} (\tau_{sst}s_t + \tau_{sjt} (H_{1t} s_t + H_{2t} x_t + F_{jt}) + \tau_{sxt}x_t + F_{st}) + M_{2t+1} x_{t+1} + F_{rt+1}$$

Collect like terms on the right.

$$(76) \quad \begin{aligned} r_{t+1} = & M_{1t+1} (\tau_{sst} + \tau_{sjt}H_{1t}) s_t \\ & + M_{1t+1} (\tau_{sxt} + \tau_{sjt}H_{2t}) x_t \\ & + M_{1t+1} (F_{st} + \tau_{sjt}F_{jt}) + M_{2t+1}x_{t+1} + F_{rt+1} \end{aligned}$$

Define the following coefficient matrices.

$$(77) \quad \mu_{1t} = M_{1t+1} (\tau_{sst} + \tau_{sjt}H_{1t})$$

$$(78) \quad \mu_{2t} = M_{1t+1} (\tau_{sxt} + \tau_{sjt}H_{2t})$$

Define the following function of time t and future exogenous variables.

$$(79) \quad G_{rt} = M_{1t+1} (F_{st} + \tau_{sjt}F_{jt}) + M_{2t+1}x_{t+1} + F_{rt+1}$$

Using these definitions, we can express a the rule for r_{t+1} in terms of s_t and x_t and future exogenous variables.

$$(80) \quad r_{t+1} = \mu_{1t}s_t + \mu_{2t}x_t + G_{rt}$$

The rule for r_t then obtains by eliminating r_{t+1} and j_t from the SSF equation for r_t , equation ??, using and use equations equation ?? and ?? respectively.

$$(81) \quad r_t = \delta_{rr} (\mu_{1t}s_t + \mu_{2t}x_t + G_{rt}) + \delta_{rs}s_t + \delta_{rj} (H_{1t}s_t + H_{2t}x_t + F_{jt}) + \delta_{rx}x_t$$

Collect like terms on the right to obtain the required rule for r_t .

$$(82) \quad r_t = (\delta_{rs} + \delta_{rr}\mu_{1t} + \delta_{rj}H_{1t}) s_t + (\delta_{rx} + \delta_{rr}\mu_{2t} + \delta_{rj}H_{2t}) x_t + \delta_{rr}G_{rt} + \delta_{rj}F_{jt}$$

The implied coefficient matrix definitions are:

$$(83) \quad M_{1t} = (\delta_{rs} + \delta_{rr}\mu_{1t} + \delta_{rj}H_{1t})$$

$$(84) \quad M_{2t} = (\delta_{rx} + \delta_{rr}\mu_{2t} + \delta_{rj}H_{2t})$$

It can be shown that these definitions are equivalent to the definitions in equations ?? and ??.

The implied function of future exogenous variables in the rule for s_t is:

$$(85) \quad F_{rt} = \delta_{rr}G_{rt} + \delta_{rj}F_{jt}$$

This definition is also equivalent to that in equation ??.

4.3 Convergence of the stable manifold

Convergence is deemed to have happened when the matrices, H_{1t} , H_{2t} , M_{1t} and M_{2t} stabilise to be independent of the time period, H_1 , H_2 , M_1 and M_2 .

As shown by ?, the stable manifold is only unique if *matrix?* has exactly one eigenvalue outside the unit circle for each costate variable in the model. Otherwise the solution is not unique. It would be helpful to be explicit about how the linearised model in this paper maps to the linear model in 1a of ? so that we can determine how to assess uniqueness based on the J matrix (J is from the Jordan canonical form of the full linear model coefficient matrix) eigenvalues when we form the linear model.

What can we say about whether convergence is guaranteed? Is there a connection between uniqueness and a guarantee of convergence?

4.4 State dynamics and system stability

To check that the stable manifold does imply stability of the system, check the eigenvalues of the matrix in the rule relating the state in period $t + 1$ to the state in period t , obtained after substituting the rules j_t and r_{t+1} back into the SSF equation for the state variables. All of those eigenvalues must lie inside the unit circle if the system is stable.

To implement this stability test and to obtain a rule for the state dynamics, it is useful to write the rule for the state variables as a function of state variables in the previous period and exogenous variables.

To do so, substitute the rule for j_t , equation ??, and the rule for r_{t+1} , equation ??, into the SSF equation for the state variables, equation ?? where t is a sufficient number of periods before T for the rules to be expressed in terms of H_1 , H_2 , M_1 and M_2 .

$$(86) \quad s_{t+1} = \delta_{sr} (\mu_{1t}s_t + \mu_{2t}x_t + G_{rt}) + \delta_{ss}s_t + \delta_{sj} (H_1s_t + H_2x_t + F_{jt}) + \delta_{sx}x_t$$

Collect like terms on the right.

$$(87) \quad s_{t+1} = (\delta_{ss} + \delta_{sr}\mu_{1t} + \delta_{sj}H_1) s_t + (\delta_{sx} + \delta_{sr}\mu_{2t} + \delta_{sj}H_2) x_t + \delta_{sr}G_{rt} + \delta_{sj}F_{jt}$$

Define the following coefficient matrices, N_1 and N_2 .

$$(88) \quad N_1 = \Delta (\delta_{ss} + \delta_{sj}H_1)$$

$$(89) \quad N_2 = \Delta (\delta_{sx} + \delta_{sj}H_2)$$

Define the function of future exogenous variables G_{st} :

$$(90) \quad G_{st} = \Delta (\delta_{sr} (M_2x_{t+1} + F_{rt+1}) + \delta_{sj}F_{jt})$$

The state transition equation can then be written as:

$$(91) \quad s_{t+1} = N_1s_t + N_2x_t + G_{st}$$

The matrix to consider when assessing stability is N_1 , the state transition matrix. Stability requires all eigenvalues of this matrix to be inside the unit circle.